## 5. Exact distribution of eigenvalues of the tridiagonal matrix

We wish to find the joint density of eigenavalues of certain random tridiagonal matrices. For this, we have to arrange the eigenvalues as a vector in $\mathbb{R}^n$, and write the density with respect to Lebesgue measure on $\mathbb{R}^n$. There are two common ways to arrange eigenvalues as a vector. Firstly, in descending order to get a vector $\lambda^{\downarrow} = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Secondly, we can place them in exchangeable random order. This means that we pick a permutation $\pi \in \mathcal{S}_n$ uniformly at random (and independently of the our random matrix), and set $\lambda_{\text{ex}} = (\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})$. Of course, if $f$ is the density of $\lambda^{\downarrow}$ and $g$ is the density of $\lambda_{\text{ex}}$, we can recover one from the other by the relationship

$$f(u_1, \dots, u_n) = n! g(u_1, \dots, u_n) \mathbf{1}_{u_1 < \dots < u_n}$$

and the fact that $g$ is symmetric in its arguments. We shall usually express the eigenvalues in exchangeable random order without explicitly saying so, but this is just a convention.

**Theorem 54.** *Let $T = T(a,b)$ be the $n \times n$ random, real symmetric matrix, with $a_k \sim N(0,1)$, $b_k^2 \sim \chi^2_{\beta(n-k)}$ and all these are independent. Then, the eigenvalues of $T$ have joint density*[4]

$$\frac{1}{\hat{Z}_{\beta,n}} \exp\left\{ -\frac{1}{4} \sum_{k=1}^{n} \lambda_k^2 \right\} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta}$$

*where the normalization constant may be explicitly found as*

$$\hat{Z}_{\beta,n} = \pi^{\frac{n}{2}} 2^{n + \frac{\beta}{4} n(n-1)} \frac{\Gamma(\frac{n\beta}{2}) \prod_{j=1}^{n-1} \Gamma(\frac{j\beta}{2})}{\Gamma\left(\frac{\beta}{2}\right)^n}.$$

**Corollary 55.** *The joint density of eigenvalues of the GOE matrix is*

$$\frac{1}{\hat{Z}_{n,1}} \exp\left\{ -\frac{1}{4} \sum_{k=1}^{n} \lambda_k^2 \right\} \prod_{i<j} |\lambda_i - \lambda_j|.$$

*The joint density of eigenvalues of the GOE matrix is*

$$\frac{1}{\tilde{Z}_{n,2}} \exp\left\{ -\frac{1}{2} \sum_{k=1}^{n} \lambda_k^2 \right\} \prod_{i<j} |\lambda_i - \lambda_j|^2$$

*where $\tilde{Z}_{n,2} = \hat{Z}_{n,2} 2^{-n^2/2}$.*

Proof of the corollary. By Theorem 49 it follows that the eigenvalues of a GOE matrix have the same distribution as the eigenvalues of the tridiagonal matrix in Theorem 54 with $\beta = 1$. This gives the first statement. The second is similar, except that there is a scaling by $\sqrt{2}$ involved in Theorem 49. ■

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[4] The corollary here was proved by by Wigner (or Dyson? before 1960 anyway) and it was noticed that the density could be generalized for any $\beta > 0$. Whether the general $\beta$-density could be realized as that of eigenvalues of a random matrix was in the air. The idea that this could be done by considering these random tridiagonal matrix with independent entries, is due to Dumitriu and Edelman. This development has had far-reaching consequences in the study of random matrices. In short, the reason is that the $\beta$-density given here is complicated to analyze, although explicit, and the tridiagonal matrix itself can be used in the analysis, as it has independent entries.

PROOF OF THEOREM 54. The joint density of $(a,b)$ on $\mathbb{R}^n \times \mathbb{R}_+^{n-1}$ is

$$\begin{aligned} f(a,b) &= \prod_{k=1}^{n} \frac{e^{-\frac{1}{4}a_k^2}}{2\sqrt{\pi}} \prod_{k=1}^{n-1} \frac{e^{-\frac{1}{2}b_k^2} b_k^{\beta(n-k)-1}}{2^{\frac{\beta}{2}(n-k)-1}\Gamma(\beta(n-k)/2)} \\ &= \frac{1}{Z_{\beta,n}} \exp\left\{-\frac{1}{4}\mathrm{tr}\left(T^2\right)\right\} \prod_{k=1}^{n-1} b_k^{\beta(n-k)-1}. \qquad (26) \end{aligned}$$

where the normalizing constant

$$Z_{\beta,n} = \pi^{\frac{n}{2}} 2^{1+\frac{\beta}{4}n(n-1)} \prod_{j=1}^{n-1} \Gamma(\beta j/2).$$

Now, let $\nu$ be the spectral measure of $T$ at the vector $\mathbf{e}_1$ (this corresponds to $\mathbf{e}_0$ of the previous section). Then $\nu = \sum_{j=1}^n p_j \delta_{\lambda_j}$. According to the previous section, $\lambda_j$ are the eigenvalues of $T$ while $p_j = |U_{1,j}|^2$ are elements of the first row of the eigenvector matrix.

Observe that almost surely none of the $b_k$s is zero, and hence by part (c) of Lemma 51, the eigenvalues of $T$ are distinct. By part (a) of the same lemma, $\mathcal{T}_n^0$ is in bijection with $\mathcal{P}_n^0$ and hence we may parameterize the matrices by $\lambda_k$, $k \le n$ and $p_k$, $k \le n-1$. We shall also write $p_n$ in many formulas, but it will always be understood to be $1 - p_1 - \ldots - p_{n-1}$. If we write $(a,b) = G(\lambda, p)$, then by the change of variable formula we get the density of $(\lambda, p)$ to be

$$\begin{aligned} g(\lambda,p) &= f(G(\lambda,p))|\det(J_G(\lambda,p))| \qquad (J_G \text{ is the Jacobian of } G) \\ &= \frac{1}{Z_{\beta,n}} \exp\left\{-\frac{1}{4}\sum_{k=1}^{n}\lambda_k^2\right\} \prod_{k=1}^{n-1} b_k^{\beta(n-k)-1} \, |\det(J_G(\lambda,p))| \qquad (27) \end{aligned}$$

It remains to find the Jacobian determinant of $G$ and express the product term in terms of $\lambda_k$ and $p_k$. For this we use the definition of spectral measure $\langle T^k\mathbf{e}_1, \mathbf{e}_1\rangle = \sum \lambda_j^k p_j$ for $k = 1, \ldots, 2n-1$. We get

$$\begin{aligned} \sum p_j\lambda_j &= T_{1,1} = a_1 & \sum p_j\lambda_j^2 &= (T^2)_{1,1} = b_1^2 + [\ldots] \\ \sum p_j\lambda_j^3 &= (T^3)_{1,1} = a_2 b_1^2 + [\ldots] & \sum p_j\lambda_j^4 &= (T^4)_{1,1} = b_2^2 b_1^2 + [\ldots] \\ \sum p_j\lambda_j^5 &= (T^5)_{1,1} = a_3 b_2^2 b_1^2 + [\ldots] & \sum p_j\lambda_j^6 &= (T^6)_{1,1} = b_3^2 b_2^2 b_1^2 + [\ldots] \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

Here the $[\ldots]$ include many terms, but all the $a_k, b_k$ that appear there have appeared in previous equations. For example, $(T^2)_{1,1} = b_1^2 + a_1^2$ and as $a_1$ appeared in the first equation, we have brushed it under $[\ldots]$.

Let $U = (u_1, \ldots, u_{2n-1})$ where $u_j = (T^j)_{1,1}$. The right hand sides of the above equations express $U$ as $F(a,b)$ while the left hand sides as $U = H(\lambda, p)$. We find the Jacobian determinants of $F$ and $H$ as follows.

<u>*Jacobian of $F$:*</u> Note that $u_{2k}$ is a function of $a_i, i \le k$ and $b_j, j \le k$ while $u_{2k-1}$ is a function of $a_i, i \le k$ and $b_j, j \le k-1$. Thus, $J_F(a,b)$ is an upper triangular matrix and we see that

$$\det(J_F(a,b)) = 2^{n-1} \prod_{k=1}^{n-1} b_k^{4(n-k)-1}. \qquad (28)$$

Jacobian of $H$: The equations above give the Jacobian of $H$ (recall that $p_n = 1 - \sum_{j=1}^{n-1} p_j$)

$$J_H(\lambda, p) = \begin{bmatrix} p_1 & \dots & p_n & \lambda_1 - \lambda_n & \dots & \lambda_{n-1} - \lambda_n \\ 2p_1\lambda_1 & \dots & 2p_n\lambda_n & \lambda_1^2 - \lambda_n^2 & \dots & \lambda_{n-1}^2 - \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ (2n-1)p_1\lambda_1^{2n-2} & \dots & (2n-1)p_n\lambda_n^{2n-2} & \lambda_1^{2n-1} - \lambda_n^{2n-1} & \dots & \lambda_{n-1}^{2n-1} - \lambda_n^{2n-1} \end{bmatrix}.$$

To find its determinant, first factor out $p_i$ from the $i^{\text{th}}$ column, for $i \le n-1$. The resulting matrix is of the same form (as if $p_i = 1$ for all $i$) and its determinant is clearly a polynomial in $\lambda_1, \dots, \lambda_n$. It must also symmetric in $\lambda_k$s, because the original problem we started with was symmetric in $\lambda_k$s (can you infer symmetry directly from the above matrix?).

If $h := \lambda_1 - \lambda_n \to 0$, then $C_{n+1} = O(h)$, $C_1 - C_n = O(h)$. Further, it is easy to check that $C_{n+1} - h(C_1 + C_2)/2 = O(h^2)$. Thus for fixed $\lambda_k$, $k \ge 2$, the polynomial in $\lambda_1$ has (at least) a four fold zero at $\lambda_n$. By symmetry, the determinant has a factor $\Delta(\lambda)^4$. However, the determinant above and $\Delta(\lambda)^4 = \prod_{i<j}(\lambda_i - \lambda_j)^4$ are both polynomials of degree $4(n-1)$. Further, the coefficient of $\lambda_1^{4n-4}$ in both is the same. Therefore we get

$$\det(J_H(a,b)) = \pm|\Delta(\lambda)|^4 \prod_{i=1}^n p_i. \tag{29}$$

From (28) and (29) we deduce that

$$|\det(J_G(\lambda, p))| = \pm \frac{\prod\limits_{i=1}^n p_i \prod\limits_{i<j} |\lambda_i - \lambda_j|^4}{2^{n-1} \prod\limits_{k=1}^{n-1} b_k^{4(n-k)-1}}.$$

Substitute this in (27) to get

$$\begin{aligned} g(\lambda, q) &= \frac{1}{2^{n-1} Z_{\beta,n}} \exp\left\{-\frac{1}{4}\sum_{k=1}^n \lambda_k^2\right\} \prod_{i=1}^n p_i \prod_{i<j} |\lambda_i - \lambda_j|^4 \left(\prod_{k=1}^{n-1} b_k^{(n-k)}\right)^4 \\ &= \frac{1}{2^{n-1} Z_{\beta,n}} \exp\left\{-\frac{1}{4}\sum_{k=1}^n \lambda_k^2\right\} \left(\prod_{i=1}^n p_i\right)^{\frac{\beta}{2}-1} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \end{aligned}$$

by part (c) of Lemma 51.

This gives the joint density of $\lambda$ and $p$ and we see that the two are independent. It remains to integrate out the $p$ variables. But that is just a Dirichlet integral

$$\int_0^1 \int_0^{1-p_1} \dots \int_0^{1-\sum_{i=1}^{n-2} p_i} \left(\prod_{i=1}^n p_i\right)^{\frac{\beta}{2}-1} dp_{n-1} \dots dp_1 = \text{Dirichlet}(\beta/2, \dots, \beta/2) = \frac{\Gamma(\beta/2)^n}{\Gamma(\beta n/2)}.$$

This completes the proof of the theorem. ∎

## 6. Beta ensembles*

Consider $n$ particles $(\lambda_1, \dots, \lambda_n)$ with density

$$g(\lambda) = \frac{1}{\hat{Z}_{\beta,n}} \exp\left\{-\frac{1}{4}\sum_{k=1}^n \lambda_k^2\right\} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

for $\beta > 0$. As we saw, this is the density of eigenvalues of random tridiagonal matrix $T_\beta$. What can we do with this density? Here are some features.

(1) *Repulsion of eigenvalues:* The density is $g(\lambda) = \exp\{-\sum V(\lambda_k)\}|\Delta(\lambda)|^\beta$ with $V(x) = x^2/4$ and where $\Delta(\lambda)$ is the Vandermonde factor. Without the Vandermonde factor (i.e., $\beta = 0$), this is the density of $n$ i.i.d variables with density $\exp\{-V(x)\}$. But $\Delta(\lambda)$ vanishes whenever $\lambda_i - \lambda_j = 0$ for some $i \neq j$. This means that the eigenvalues tend to keep away from each other. Further, the vanishing of $|\lambda_i - \lambda_j|^\beta$ increases with $\beta$ which means that the repulsion increases with $\beta$. As $\beta \to \infty$, the density concentrates at a particular configuration, or "the particles freeze at the lowest energy configuration".

(2) *Gibbs interpretation of the density:* For convenience, scale the eigenvalues down by $\sqrt{\beta}$. Continue to denote the variables by $\lambda_k$. The resulting density is $f_\beta(\lambda) = g_\beta(\lambda\sqrt{\beta}) = \exp\{-\beta H_{n,\beta}(\lambda)\}$ where

$$H_n(\lambda) = \frac{1}{4}\sum_{k=1}^{n} \lambda_k^2 - \frac{1}{2}\sum_{i \neq j} \log|\lambda_i - \lambda_j|.$$

$H_n(\lambda)$ is called the energy of the configuration $\lambda$. According to Boltzmann, all systems in Statistical mechanics have this structure - the density is $\exp\{-\text{energy}\}$ where the energy (or *Hamiltonian*) varies from system to system and in fact characterizes the system.

In the case at hand, the energy has two terms. The function $V$ is interpreted as a potential, a particle sitting at a location $x$ will have potential energy $V(x)$. Further, there is pairwise interaction - if a particle is at location $x$ and another at $y$, then they have an interaction potential of $-\log|x-y|$. This just means that they repel each other with force (which is the gradient of the interaction energy) $1/|x-y|$ (repulsion rather than attraction, because of the negative sign on $\log|x-y|$). This is precisely Coulomb's law, suitably modified because we are not in three dimensions. More physically, if one imagines infinite sheets of uniformly charged plates placed perpendicular to the x-axis, and a potential $V(x)$ is applied, then they repel each other by a force that is inverse of the distance.

Thus, they prefer to locate themselves at points $x_1, \ldots, x_n$ that minimizes the energy $H_n(x)$. However, if there is a positive temperature $1/\beta$, then they don't quite stabilize at the minimum, but have a probability to be at other locations, but with the density that decreases exponential with the energy. Thus the density is given exactly by the density $g_\beta(\lambda)$! This is called a *one-component plasma* on the line.

(3) Note that we ignored the normalization constants in the previous discussion. Many probability distributions that arise in probability are described by giving their density as $Z_\beta^{-1}\exp\{-\beta H(x)\}$ where $H(\cdot)$ is specified. The trouble is analyzing the system to make useful statements about a typical configuration sampled from this measure. As $Z_\beta = \int \exp\{-\beta H(x)\}dx$, we see that $Z_\beta$ is like a Laplace trnaform of the function $H(x)$. Thus, if we can compute $Z_\beta$ (for all $\beta$), one can deduce many things about the distribution. For example, the expected energy of a random sample from the given density is

$$\frac{1}{Z_\beta}\int H(x)\exp\{-\beta H(x)\}dx = \frac{1}{Z_\beta}\frac{\partial Z_\beta}{\partial \beta} = \frac{\partial}{\partial \beta}\log Z_\beta.$$

This is the reason why physicists lay great stress on finding the normalization constant $Z_\beta$, which they term the *partition function*. Generally speaking, computing $Z_\beta$ is fairly impossible. The system that we have, the one with energy function $H_n$, is exceptional in that the partition function can be found explicitly, as we did in the previous section!

(4) The computation of the normalization constant from the previous section proves the following highly non-trivial integration formula (try proving it!)

$$\int_{\mathbb{R}^n} \exp\left\{-\frac{1}{4}\sum_{k=1}^{n}\lambda_k^2\right\}\prod_{i<j}|\lambda_i-\lambda_j|^{\beta}d\lambda_1\dots d\lambda_n = \pi^{\frac{n}{2}}2^{n+\frac{\beta}{4}n(n-1)}\frac{\Gamma(\frac{n\beta}{2})\prod_{j=1}^{n-1}\Gamma(\frac{j\beta}{2})}{\Gamma\left(\frac{\beta}{2}\right)^n}. \tag{30}$$

This can be derived from a similar but more general integral of Selberg, who computed

$$S(\alpha,\beta,\gamma)=\int_{[0,1]^n}|\Delta(x)|^{2\gamma}\prod_{i=1}^{n}x_i^{\alpha-1}(1-x_i)^{\beta-1}dx$$

where $\Delta(x)=\prod_{i<j}|x_i-x_j|$ and $\alpha,\beta,\gamma$ are complex parameters satisfying some inequalities so that the integral converges[5].

But this does not cover the main questions one would like to answer when an explicit density $g(\lambda_1,\dots,\lambda_n)$ is at hand. Observe that the labeling here is introduced for convenience, and what we care about is the empirical measure $L_n = n^{-1}\sum_{k=1}^{n}\delta_{\lambda_k}$. If $\lambda$ has density $g_\beta(\lambda)$, what is $\mathbf{E}[L_n[a,b]]$ for any $a<b$? What about the variance $\mathrm{Var}(L_n[a,b])$? What is the typical spacing between one eigenvalue and the next? What is the chance that there in no eigenvalue in a given interval? Does $L_n$ (perhaps after rescaling $\lambda_k$) converge to a fixed measure (perhaps the semicircle law) as $n\to\infty$?

The last question can actually be answered from the joint density, but the other questions are more "local". For example, if $I=[a,b]$, then by the exchangeability of $\lambda_k$s

$$\mathbf{E}[L_{n,\beta}(I)] = n\mathbf{P}(\lambda_1\in I) = n\int_I\left(\int_{\mathbb{R}^{n-1}} g_\beta(\lambda_1,\dots,\lambda_n)d\lambda_n\dots d\lambda_2\right)d\lambda_1$$

which involves integrating out some of the variables. Can we do this explicitly? It is not clear at all from the density $g_\beta$. In fact, there is no known method to do this, except for special values of $\beta$, especially $\beta=1,2,4$. Of these $\beta=2$ is particularly nice, and we shall concentrate on this case in the next few sections.

[5]More on the Selberg integral, its proofs and its consequences may be found in the book of Mehta or of Andrews, Askey and Roy.